Descriptive Set Theory HW 4

Thomas Dean

Problem 1. Let X be a topological space and $A \subseteq X$.

1. Show that $U(A)$ is regular open.

2. If moreover X is a Baire space and A has the BP, then $U(A)$ is the unique regular open set U with $A =^* U$.

Solution.

1. By definition, $U(A) = \bigcup\{U : U \Vdash A\}$ is open. Since $U(A) \subseteq \overline{U(A)}$, we have that $U(A) \subseteq \text{int}(\overline{U(A)})$. Going the other direction, first recall that $\overline{U} \setminus U$ and $F \setminus \text{int}(F)$ are closed nowhere dense, where U and F are arbitrary open and closed sets, respectively. From these two, we get by definition that $U(A) =^* \overline{U(A)} =^* \text{int}(\overline{U(A)})$. From Anush's notes we have that $U(A) \Vdash A$, and so we may conclude using the above sentence that $\text{int}(\overline{U(A)}) \Vdash A$.

   To justify this further, we have more generally that if $B =^* C$ and $B \Vdash A$, then $C \Vdash A$, where $B, C \subseteq X$ are arbitrary. The reason is since $C \setminus A \subseteq (C \setminus B) \cup (B \setminus A)$. Anyways, since $\text{int}(\overline{U(A)}) \Vdash A$, we get $\text{int}(\overline{U(A)}) \subseteq U(A)$ by definition of $U(A)$. So, $\text{int}(\overline{U(A)}) = U(A)$, implying $U(A)$ is regular open.

2. By (1), we have that $U(A)$ is regular open. Further, since A has the BP, Anush's notes give us that $A =^* U(A)$. Assume now that $V =^* A$ for some regular open V. We have immediately that $V \Vdash A$, and so $V \subseteq U(A)$. Since V is regular open, assume for the sake of contradiction that there is some $x \in U(A) \setminus \text{int}(\overline{V})$. Since $U(A)$ is open, we have that $U(A) \not\subseteq \overline{V}$. So, we may fix $y \in U(A) \setminus \overline{V}$. Since $y \notin \overline{V}$, we may fix a (nonempty) open neighborhood W of y such that $W \cap V = \varnothing$. By intersecting with $U(A)$, we may assume that $W \subseteq U(A)$. However, by hypothesis, we have that $U(A) =^* A =^* V$. This implies that $W \subseteq U(A) \setminus V$ is meager, contradicting that X is a Baire space.

⋆

Problem 2. Let X be a Baire space. Recall that if $\Gamma \curvearrowright X$ acts by homeomorphism, we say it is called generically ergodic if every invariant set $A \subseteq X$ with the BP is either meager or comeager. Prove that the following are equivalent:

1. $\Gamma \curvearrowright X$ is generically ergodic.
2. Every invariant nonempty open set is dense.
3. For comeager-many $x \in X$, the orbit $[x]_\Gamma$ is dense.
4. There is a dense orbit.
5. For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \varnothing$.

Solution.

(1) $\Rightarrow$ (2) : If U is an invariant nonempty open set, then by (1) we have that U is either meager or comeager. Since X is a Baire space, U must be comeager. But then we win because comeager subsets of a Baire space are dense.

(2) $\Rightarrow$ (3) : We first begin with a claim.

Claim 1. If U is open, so is $[U]_\Gamma$.

Proof. For each $\gamma \in \Gamma$, the map $f \colon x \mapsto \gamma.x$ is a homeomorphism. So, $\gamma U = f"U$ is open. Then $[U]_\Gamma = \bigcup_\gamma \gamma U$ is open. □

Following Anush's hint, fix a countable basis $\{U_n\}_{n \in \mathbb{N}}$ of nonempty U_n and consider $U = \bigcap_n [U_n]_\Gamma$. By the claim, each $[U_n]_\Gamma$ is open. By (2), $[U_n]_\Gamma$ is dense, as it's invariant by definition. Observe, U is comeager as it's a dense G_δ. It's enough to show every $x \in U$ has dense orbit. Fix $x \in U$ and a nonempty V. So, there's some $U_n \subseteq V$. Since $x \in [U_n]_\Gamma$, there's a $\gamma \in \Gamma$ and $u \in U_n$ such that $x = \gamma.u$, implying that $\gamma^{-1}.x = u \in V$. Then $\gamma^{-1}.x \in [x]_\Gamma \cap V$, implying that $[x]_\Gamma$ is dense for comeagerly many x.

(3) $\Rightarrow$ (4) : Since X is Baire, the result is immediate.

(4) $\Rightarrow$ (5) : Fix a dense orbit $[x]_\Gamma$ and two nonempty open U, V. Let $g, h \in \Gamma$ be such that $g.x \in U$ and $h.x \in V$ If $\gamma = hg^{-1}$, then $h.x = \gamma.(g.x) \in (\gamma U) \cap V$.

(5) $\Rightarrow$ (1) : Assume that A has the BP, is invariant, but isn't meager or comeager. In particular, both A and A^c aren't meager, implying that there are nonempty $U, V \subseteq X$ such that $U \Vdash A$ and $V \Vdash A^c$. By (5), there's a $\gamma \in \Gamma$ such that $W = (\gamma U) \cap V \neq \varnothing$. Since $U \Vdash A$ and the map $x \mapsto \gamma.x$ is a

homeomorphism, we get that $\gamma U \Vdash \gamma A$. Since A is invariant, we have $\gamma U \Vdash A$. Since $W \subseteq \gamma U$ and $W \subseteq V$, we get that $W \Vdash A$ and $W \Vdash A^c$, implying that W is meager. This contradicts that X is Baire. ⋆

Problem 3. Show that the Kuratowski-Ulam theorem fails if A does not have the BP by constructing a non-meager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line.

Solution. There are continuum-many Borel subsets of $\mathbb{R}^2$, and so continuum many meager F_σ sets. Let $(F_\xi)_{\xi<\mathfrak{c}}$ be an enumeration of the meager F_σ sets. We construct a sequence $(a_\xi)_{\xi<\mathfrak{c}}$ of points in $\mathbb{R}^2$ such that $\{a_\lambda \colon \lambda \leq \xi\} \not\subseteq F_\xi$ for each $\xi < \mathfrak{c}$, and no three points of A are on a straight line.

If we let $A = \{a_\lambda \colon \lambda < \mathfrak{c}\}$, then A isn't meager because it won't be a subset of any meager F_σ set. Let's first show why this construction is possible, then explain how this implies A doesn't have the BP because Kuratowski-Ulam fails for A.

Assume that we've defined $(a_\lambda)_{\lambda<\xi}$ and (1)-(2) hold for all $\theta < \xi$. Let $B = \{a_\lambda \colon \lambda < \xi\}$. Notice that $|B| < \mathfrak{c}$. For each $\alpha, \beta < \xi$, let $L_{\alpha,\beta}$ denote the unique line connecting a_α and a_β. Since $|B|^2 < \mathfrak{c}$, observe that $|\{L_{\alpha,\beta} \colon \alpha, \beta < \xi\}| < \mathfrak{c}$.

Keeping the above in the back of our minds for a second, notice that since F_ξ^c is comeager, we get that $|F_\xi^c| = \mathfrak{c}$. To see why, observe that F_ξ^c contains a dense G_δ, say G. Since $\mathbb{R}^2$ is perfect, it's not too hard to show G is perfect as well. But then G is a nonempty perfect Polish space, and therefore has cardinality $\mathfrak{c}$.

Since $|F_\xi^c| = \mathfrak{c}$ and $|\{L_{\alpha,\beta} \colon \alpha, \beta < \xi\}| < \mathfrak{c}$, we may choose $a_\xi \in F_\xi^c \setminus \bigcup\{L_{\alpha,\beta} \colon \alpha, \beta < \xi\}$. The reason we may do this is that, otherwise, $F_\xi^c \subseteq L = \bigcup\{L_{\alpha,\beta} \colon \alpha, \beta < \xi\}$. This implies that L is comeager. Comeager things have BP, so by Kuratowski-Ulam, we have that L_x is comeager for comeagerly many $x \in \mathbb{R}$. So, fix some a witnessing this. Now, $|L_a| < \mathfrak{c}$ because $|L_{\alpha,\beta} \cap V_a| \leq 1$, where V_a is the vertical line $x = a$. In other words, each $L_{\alpha,\beta}$ adds at most one element to L_a, and there are $|\xi^2| < \mathfrak{c}$ many such lines. This contradicts the above paragraph that comeager subsets of perfect Polish spaces have cardinality continuum. This completely the construction. Observe that by the construction, no three points of A are on the same line and A is not meager.

Finally, assume that A has the BP. Then Kuratowski-Ulam implies that $\neg\forall^* x(A_x \text{ is meager}) \Leftrightarrow \exists^* x(A_x \text{ is not meager})$. In particular, fix $x_0 \in \mathbb{R}$ such that A_{x_0} is not meager. Since A_{x_0} is not meager, it has at least three elements, say a, b, and c. But then, $(x_0, a), (x_0, b)$, and (x_0, c) are all in A and all lie on the vertical line $x = x_0$. This contradiction gives the result. ⋆

Problem 4. Show that if X, Y are second countable Baire Spaces, then so is $X \times Y$.

Solution. The product is second countable, so it's enough to show that it's Baire. Towards that end it's enough to show $U \times V$ is not meager, where U, V are nonempty open sets. If instead that $U \times V$ is meager, then Kuratowski Ulam implies that $\forall^* x((U \times V)_x$ is meager). Since X is Baire, we have that $\exists^* x(x \in U)$. This implies that $\exists^* x(x \in U$ and $(U \times V)_x$ is meager). If we fix $x \in U$ witnessing this, then this implies that $(U \times V)_x = V$ is meager, contradicting that Y is Baire. ⋆

Problem 5. Let $X = [0,1]^\omega$. Show that $C_0 = \{(x_n) \in X \colon (x_n) \to 0\}$ is $\mathbf{\Pi}^0_3$.

Solution. Recall that $(x_n) \to 0 \iff (\forall n \in \mathbb{N})(\exists N \in \mathbb{N})(\forall k \geq N)|x_k| \leq \frac{1}{n}$. For a fixed $k, n \in \mathbb{N}$, notice that $U_{k,n} = \{(x_i)_{i<\omega} \in X \colon |x_k| \leq \frac{1}{n}\}$ is closed, as $U_{k,n} = \mathrm{proj}_k^{-1}[[0, \frac{1}{n}]]$ and $[0, \frac{1}{n}]$ is relatively closed in $[0,1]$. But, then

$$C_0 = \bigcap_{n<\omega} \bigcup_{N<\omega} \bigcap_{k \geq N} U_{n,k}$$

is $\mathbf{\Pi}^0_3$ as desired. ⋆

Problem 6. Show that if $\mathbf{\Gamma}$ is a self dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set $\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$ can have X-universal sets.

Solution. The final sentence is clear because $\mathcal{B}(X)$ of Borel sets and the classes $\mathbf{\Delta}^0_\xi(X)$ both satisfy the hypothesis of the problem. Assume towards a contradiction that $U \subseteq X \times X$ is X-universal for $\mathbf{\Gamma}(X)$. Since $U \in \mathbf{\Gamma}(X \times X)$, we have that $U^c \in \neg\mathbf{\Gamma}(X \times X) = \mathbf{\Gamma}(X \times X)$. The map $d \colon x \mapsto (x, x)$ is continuous, so we have that $V = d^{-1}[U^c] = \{x \in X \colon (x, x) \notin U\} \in \mathbf{\Gamma}(X)$, because $\mathbf{\Gamma}$ is closed under continuous preimages. This is a contradiction, because V is the antidiagonal of U, which cannot be equal to U_x for any $x \in X$. ⋆

Problem 7. Let X, Y be topological spaces and let $\mathrm{proj}_X \colon X \times Y \to X$ be the projection function. Prove the following statements:

1. proj_X is continuous and open.
2. proj_X does not in general map closed sets to closed sets, even for $X = Y = \mathbb{R}$.
3. For $X = Y = \mathbb{R}$ (or in general any σ-compact Hausdorff space), proj_X maps closed sets to σ-compact (and hence F_σ) sets.
4. If Y is compact, then proj_X indeed maps closed sets to closed sets.

Solution.

1. For an open $U \subseteq X$, $\mathrm{proj}_X^{-1}[U] = U \times Y$ is open. So proj_X is continuous. Next, given nonempty open $U \subseteq X$ and $V \subseteq Y$, $\mathrm{proj}_X[U \times V] = U$ is open. If one factor is empty, then the projection is empty. It follows proj_X is open.
2. Let $C = \{(x, y) \in \mathbb{R}^2 \colon y = \frac{1}{x}, 0 < x \leq 1\}$. This is closed, but $\mathrm{proj}_X{}''C = (0, 1]$ isn't closed.
3. If X, Y are σ-compact Hausdorff spaces, let $X = \bigcup_n K_n$ and $Y = \bigcup_n L_n$ where each K_n, L_n are compact. Now, assume that we have a closed $C \subseteq X \times Y = \bigcup_{n,m}(K_n \times L_m)$. Then $C = \bigcup_{n,m}(C \cap (K_n \times L_m))$. For each $n, m < \omega$, $C_{n,m} = C \cap (K_n \times L_m)$ is compact, because it's a closed subset of the compact space $K_n \times L_m$. Since proj_X is continuous, $\mathrm{proj}_X{}''C_{n,m} \subseteq X$ is compact. Since X is Hausdorff, $\mathrm{proj}_X{}''C_{n,m}$ is closed. Then, $\mathrm{proj}_X{}''C = \bigcup_{n,m} \mathrm{proj}_X{}''C_{n,m}$ is F_σ (and σ-compact).
4. For notational simplicity, denote proj_X by π. Let $C \subseteq X \times Y$ be closed. Fix $x \in X \setminus \pi''C$. So, $(x, y) \notin C$ for each $y \in Y$. For each $y \in Y$, fix U_y, V_y open such that $(x, y) \in U_y \times V_y \subseteq (X \times Y) \setminus C$. Then $\{V_y \colon y \in Y\}$ is an open cover of Y. By compactness, there's a finite subcover $\{V_{y_i} \colon 1 \leq i \leq n\}$. Consider the open set $U = \bigcap_{i \leq n} U_{y_i}$. By construction, $x \in U$. It is enough to show that $U \subseteq X \setminus \pi''C$. Otherwise, fix $a \in U \cap \pi''C$ and fix $y \in Y$ such that $(a, y) \in C$. So, $y \in V_{y_i}$ for some $i \leq n$. Then, $(a, y) \in U \times V_{y_i} \subseteq (U_{y_i} \times V_{y_i}) \subseteq (X \times Y) \setminus C$, a contradiction. So $\pi''C$ is closed by definition.

⋆

Problem 8. Prove the following:

1. Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets.

2. Let (X, τ), (Y, σ) be Polish and $f: X \to Y$ a Borel isomorphism. Show that there are Polish topologies $\tau_X \supseteq \tau$, $\sigma_Y \supseteq \sigma$ with the same Borel sets as before such that $f: (X, \tau_X) \to (Y, \sigma_Y)$ is a homeomorphism. Moreover, τ_X and σ_Y can be taken to be zero dimensional.

Solution.

1. Let (X, τ) be a Polish space with $\mathcal{U} = \{U_n : n < \omega\}$ a countable base. For some collection $\mathcal{A}$ of subsets of X, let $T(\mathcal{A})$ denote the smallest topology containing all sets in $\mathcal{A}$. We first start with a claim.

   Claim 2. If $T(\mathcal{A}_i) = \mathcal{B}_i$ for each $i \in I$, then $T(\bigcup_i \mathcal{B}_i) = T(\bigcup_i \mathcal{A}_i)$.

   Proof. $T(\bigcup_i \mathcal{B}_i) \supseteq T(\bigcup_i \mathcal{A}_i)$ is clear. Going the other direction, note that $\mathcal{B}_i = T(\mathcal{A}_i) \subseteq T(\bigcup_i \mathcal{A}_i)$ for each $i \in I$. So, $\bigcup_i \mathcal{B}_i \subseteq T(\bigcup_i \mathcal{A}_i)$. Then, $T(\bigcup_i \mathcal{B}_i) \subseteq T(T(\bigcup_i \mathcal{A}_i)) = T(\bigcup_i \mathcal{A}_i)$. □

   A similar argument shows that $T(T(\mathcal{B}) \cup \mathcal{D}) = T(\mathcal{B} \cup \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{P}(X)$. For each $n < \omega$, let $\tau_n = T(\tau \cup \{X \setminus U_n\}) = T(\mathcal{U} \cup \{X \setminus U_n\})$ be the Polish topology refining τ making U_n clopen. Then, by Anush's notes and the claim, $\tau_\infty = T(\bigcup_n \tau_n) = T(\mathcal{U} \cup \{X \setminus U_n : n < \omega\})$ is a Polish topology refining τ with the same Borel sets as τ. A basis for τ_∞ would be the collection of all finite intersections of elements of $\mathcal{U} \cup \{X \setminus U_n : n < \omega\}$, and since all elements of this collection are τ_∞-clopen, we get that (X, τ_∞) is zero-dimensional, as desired.

2. We construct sequences $(\tau_n : n < \omega)$ and $(\sigma_n : n < \omega)$ each with the same Borel sets as τ and σ in the following way: Let $\tau_0 = \tau$ and $\sigma_0 = \sigma$. Given τ_n and σ_n, we may use Anush's notes to first find a topology $\tau^* \supseteq \tau$ such that $f: (X, \tau^*) \to (Y, \sigma_n)$ is continuous. Then, we may use the above problem to choose $\tau_{n+1} \supseteq \tau^*$ to be a zero-dimensional Polish topology. Notice in particular that $f: (X, \tau_{n+1}) \to (Y, \sigma_n)$ is still continuous. Next, observe that $f^{-1}: (Y, \sigma_n) \to (X, \tau_{n+1})$ is Borel because τ_{n+1} and τ have the same Borel sets by construction. So, we may similarly refine $\sigma^* \supseteq \sigma_n$ to make f^{-1} continuous, and the refine again to find a zero-dimensional Polish topology $\sigma_{n+1} \supseteq \sigma$ with the same Borel sets as σ. Like before,

note that $f^{-1}\colon (Y, \sigma_{n+1}) \to (X, \tau_{n+1})$ is still continuous This completes the construction.

Next, let $\tau_X = T(\bigcup_n \tau_n)$ and $\sigma_Y = T(\bigcup_n \sigma_n)$. By Anush's notes, these are both Polish topologies with the same Borel sets as τ and σ. Further they are zero-dimensional: for example, if $\mathcal{U}_n$ is a clopen basis for τ_n, then the claim above implies that $\tau_X = T(\bigcup_n \mathcal{U}_n)$. Like the previous problem, a basis for τ_X would be the collection of finite intersections of elements in $\bigcup_n \mathcal{U}_n$. But since each element of any $\mathcal{U}_n$ would remain τ_X-clopen, we get that this basis is of τ_X-clopen sets.

Finally, we check that f is now a homeomorphism. For this it's enough to check that $f^{-1}[V] \in \tau_X$ for any $V \in \bigcup_n \sigma_n$ and that $f"U = (f^{-1})^{-1}[U] \in \tau_Y$ for any $U \in \bigcup_n \tau_n$. Given $V \in \sigma_n$ for some n, we have by construction that $f^{-1}[V] \in \tau_{n+1} \subseteq \tau_X$, because $f\colon (X, \tau_{n+1}) \to (Y, \sigma_n)$ is continuous. Similarly, given $U \in \tau_n$ for some n, we have that $U \in \tau_{n+1}$ because τ_{n+1} refines τ_n. Then, by construction we have that $(f^{-1})^{-1}[U] \in \sigma_{n+1} \subseteq \sigma_Y$ because $f^{-1}\colon (Y, \sigma_{n+1}) \to (X, \tau_{n+1})$ is continuous.

⋆